International Journal of Theoretical Physics, Vol. 26, No. 2, 1987

Non-Hermitian Operators, Localization Problem, and the Conservation of Fermion Number

Pratul Bandyopadhyay[1]

Received April 3, 1984

The localization problem of relativistic particles is studied, considering that an elementary spin-$\frac{1}{2}$ particle of nonvanishing mass actually corresponds to an extended body. The role of non-Hermitian operators in describing such a system is discussed and it is pointed out that the localization of such a system is linked to the global conservation of fermion number given by the "internal helicity" of such a system. The localization region is then determined by means of imprimitivity systems for particular representations of the de Sitter group $SO(4,1)$.

1. INTRODUCTION

The localization problem of relativistic particles is well known in quantum physics. The problem stems from the fact that the position operator is non-Hermitian in this case. Many authors have tried to solve the problem by constructing position operators in such a way that this may be made compatible with the well-known properties of quantum mechanics. However, none of these approaches are beyond ambiguity. Newton and Wigner (1949) in their classical work showed that for a single particle the notion of localizability (whenever it exists) is uniquely determined by relativistic kinematics. But in that work and later in a mathematically more rigorous reformulation by Wightman (1962) it was pointed out that the position operator exists for particles with nonvanishing masses and arbitrary masses but does not exist for massless particles realized in nature. Indeed, the trouble with Newton–Wigner solution is that it violates the Lorentz invariance of localization. This directly excludes massless particles from its purview. But if we take into account that Lorentz invariance is the basic

[1]Indian Statistical Institute, Calcutta-700035, India.

0020-7748/87/0200-0131$05.00/0

canon of any relativistic system as it is generally considered, Newton–Wigner treatment of massive particles also seems to be improper.

The localization of massless particles has been studied by Bayen and Niederle (1981), by replacing the Poincaré group by the conformal group, $R^2 \otimes E(2)$ by the de Sitter group $SO_0(4, 1)$ as suggested by Angelopoulos et al. (1974), and introducing the concept of localizability by means of imprimitivity systems for the particular representations of the de Sitter group $SO_0(4, 1)$. They have shown that massless particles having spin <2 can indeed be localized in a region topologically equivalent to a 3-sphere. However, since for a particle of nonvanishing mass conformal interpretation is not possible, it appears that this approach is unsuitable for such systems.

Kalnay (1970, 1971), in a series of papers, has tried to construct a position operator for a massive as well as for a massless particle in a unified way incorporating Lorentz invariance. However, the generalized position operator prescribed by him is found to be non-Hermitian and even nonnormal. But he has emphatically argued that non-Hermitian operators should not be rejected outright, since they may give more information than the usual Hermitian operators. Indeed, he has pointed out, in general non-Hermitian operators have eigenvalues that are complex numbers, which may be thought of as an ordered pair of real numbers. This extra variable may lead to some intrinsic property of the particles. In view of this, Kalnay and Toledo (1967) have considered that a relativistic particle may not be a point particle and the notion of localization should be modified such that the values of the positions be certain regions. Non-Hermitian operators are suitable in this approach because to each coordinate a pair of variables is associated that may denote some property of the extended body in addition to the location of the center of the region.

In this paper, we try to show that the localization of relativistic massive particles with spatial extension can be achieved in the framework suggested by Angelopoulos, et al (1974) and developed by Bayen and Niederle (1981) for massless particles, extending the Poincaré group where we have an extra constraint, such as the global conservation of fermion number, which is uniquely satisfied. It is noted that for massless particles, the fermion number is given by the helicity states of a spin-$\frac{1}{2}$ particle and hence no extra constraint is necessary. But for massive particles, the extended nature of the body allows us to have the "internal helicity" or orientation of the system, which is identified with the fermion number. Once the localization region is achieved, the position operator can be defined as a non-Hermitian operator where the pair of variables of the complex eigenvalue can be associated with the center of the region and the fermion number, which is a constant of motion. This formalism, however, compels us to consider all bosons as composite states.

2. AXIOMS OF LOCALIZABILITY AND NON-HERMITIAN OPERATORS

Newton and Wigner (1949) for the proper definition of the position of a particle the following axioms.

(a) The set S_0 of states localized at the origin of space-time is a linear set.

(b) The set is invariant under spatial rotations about the origin 0 and under space-time reflection.

(c) States localized at $(t=0, x)$ and $(t=0, x')$ with $x \neq x'$ are orthogonal.

(d) A condition of mathematical good behavior holds.

On the basis of the above assumptions, Newton and Wigner deduced a unique position operator, which for spin zero takes the form in the p-representation, $t=0$,

$$X_{\mathrm{NW}} = i\left(\frac{\partial}{\partial \mathbf{p}}\right) - \frac{i\mathbf{p}}{2E_p^2} \tag{1}$$

The 3-localized states in $(t=0, \mathbf{a})$ i.e., the eigenfunctions in

$$X_{\mathrm{NW}} \phi_{\mathbf{a}}^{\mathrm{NW}} = \mathbf{a} \phi_{\mathbf{a}}^{\mathrm{NW}} \tag{2}$$

are given by

$$\phi_{\mathbf{a}}^{\mathrm{NW}} = \frac{1}{(2\pi)^{3/2}} [\exp(-i\mathbf{a} \cdot \mathbf{p})] E_p^{1/2} \tag{3}$$

As mentioned earlier, this solution violates the Lorentz invariance of localization. This has prompted several authors to generalize the basic axioms. Note that axiom (c) expresses the fact that the probability of finding the particle at $t=0$ at the point x is zero if it is certainly at $x' \neq x$ at the same time. Evidently this is not necessary if the particle has some spatial extension. Related to this is the fact that the hermiticity of x may be abandoned.

Kalnay (1970, 1971) has developed a position operator that is applicable to both massive and massless particles and which can be put in the momentum representation, Heisenberg picture, $t=0$, as follows:

$$X = \Lambda(i\partial/\partial \mathbf{p})\Lambda \tag{4}$$

where Λ is the corresponding projection operator in the allowed vector space. The position operator (4) is non-Hermitian and even nonnormal, but its eigenvalues are real. However, this can be analysed as a binary variable, as is the case with all non-Hermitian operators, and in that case its first part is equal for spinless massive mesons to Newton–Wigner's operator, while the second part is a constant of the motion.

This aspect suggests that the second part of the binary variable may lead us to an intrinsic property of the particle. Indeed, when this non-Hermitian operator is taken to represent an extended region, the first part of the binary variable corresponds to the location of a center of the region and the second part is related to some intrinsic property. This intrinsic property, we show in the following section, is related to the fermion number of a spin-$\frac{1}{2}$ particle. Thus, we see that Kalnay's generalized non-Hermitian operator (5) can be taken to correspond to the location of the center of the region and a quantum number that is a constant of motion.

In case the particle is localized in a region Δ of a space S, a projection operator $E(\Delta)$ acting in a Hilbert space $\mathscr{H}$ can be defined, which obeys the following axioms (Wightman, 1962; Bayen and Niederle, 1982):

1. For every Borel set $\Delta \subset S$, there is a projection operator $E(\Delta)$, the expectation value of which is the probability of finding the particle in Δ.
2. $E(\Delta_1 \cap \Delta_2) = E(\Delta_1) \cdot E(\Delta_2)$.
3. $E(\Delta_1 \cup \Delta_2) = E(\Delta_1) + E(\Delta_2) - E(\Delta_1 \cap \Delta_2)$.
4. $E(S) = 1$.
5. $E(g\Lambda) = U(g)E(\Delta)U^{-1}(g), \quad g \in SO(4, 1)$.

Here U is a unitary operator from the representation of the de Sitter group $SO(4, 1)$.

Any family of such projection operators $\{E(\Delta)\}$ uniquely determines the position operator q_k via the spectral decomposition

$$q_k = \int_{-\alpha}^{\alpha} \lambda \, dE \qquad (\{\lambda_k \leq \lambda\}) \tag{5}$$

Note that any function $\Delta \to E(\Delta)$ from the Borel sets to the projection operator $E(\Delta)$ satisfying the above axioms froms a system of imprimitivity for the representation of the group $SO(4, 1)$. Utilizing this property, we study in the following section the topological properties of the localization region of a relativistic massive particle, which also gives rise to the fermion number as a good quantum number.

3. LOCALIZATION REGION AND THE CONSERVATION OF FERMION NUMBER

An extended body can be acted upon by the de Sitter group $SO(4, 1)$. Indeed, the wave function of the form $\psi(x_\mu, \xi_\mu)$, where ξ_μ is an internal variable attached to each point in space-time, gives rise to dilatation in addition to Poincaré transformation and thus may be taken to depict an extension of the internal space. When ξ_μ is identified with a "direction vector" or "velocity vector," this may be taken to correspond to an "internal helicity." Roman et al. (1972) have shown that the inclusion of this extra variable extends the Poincaré group $SO(3, 1)$ to the de Sitter group $SO(4, 1)$.

The irreducible representations of $SO(4)$ are characterized by two numbers (k_0, n), where $|k_0|$ is integer or half-integer and n is a natural number. These two numbers are related to the values of the Casimir operators by

$$\begin{aligned} \tfrac{1}{2}S_{\alpha\beta}S^{\alpha\beta} &= k_0^2+(|k_0|+n)^2-1 \\ \tfrac{1}{8}\varepsilon^{\alpha\beta\gamma\delta}S_{\alpha\beta}S_{\gamma\delta} &= k_0(|k_0|+n) \end{aligned} \tag{6}$$

where $S_{\alpha\beta}$, α, $\beta = 1, 2, 3, 4$, are the generators of the group. The irreducible representations of $SO(4, 1)$ have been investigated by Dixmier (1961). Barut and Bohm (1970) have shown that the representation of $SO(4, 1)$ given by $k_0 = 1/2$ and $-1/2$ can be extended to two inequivalent representations of $SO(4, 2)$. In fact, these k_0 values actually correspond to the eigenvalues of the operator $K_0 = 1/2(a^+a - b^+b)$ in the oscillator representation of the $SO(3)^1 \otimes SO(3)^2$ basis of $SO(4, 1)$. Barut and Bohm have shown that no other representations except those corresponding to the eigenvalues $k_0 = +1/2$ and $-1/2$ apart from the trivial case $k_0 = 0$ can be fully extended to $SO(4, 2)$. Besides, the value of k_0 as well as its signature is an $SO(4, 2)$ invariant.

The representation $(s = 0,\ k_0 = 0)$ in the conformal interpretation of $O(4, 2)$ describes massless spin-0 particles. The representation $(s = 1/2,\ k_0 = \pm 1/2)$ describes the helicity states of a massless spinor. Now, for a massive particle, the conformal invariance breaks down, so that $k_0 = \pm 1/2$ cannot be interpreted as helicity states in the conventional sense. To find the relevance of these states for a massive particle, we note that an $O(4, 2)$ spinor is given by an eight-component spinor, which may be split into two four-component spinors with a certain "orientation." Indeed, the doublet of four component spinors

$$\begin{vmatrix} \phi_1 \\ \phi_2 \end{vmatrix}$$

representing an $O(4, 2)$ spinor ξ is characterized by the fact that the space, time, or conformal reflection changes $\phi_1 \rightleftarrows \phi_2$. In Minkowski space, they satisfy the coupled equations

$$i\partial\!\!\!/\phi_1 = -m\phi_2, \qquad i\partial\!\!\!/\phi_2 = -m\phi_1 \tag{7}$$

However, it is also possible to obtain a pair of standard Dirac spinors in Minkowski space

$$\begin{vmatrix} \varphi_1 \\ \varphi_2 \end{vmatrix}$$

to represent the conformal spinor such that only conformal reflection changes, $\varphi_1 \rightleftarrows \varphi_2$. Now, the coupled equations (8) do not allow the Cartan semispinor doublets

$$\begin{vmatrix} \phi_1 \\ \phi_2 \end{vmatrix}$$

to be physically observable unless $m=0$. However, if we define ϕ_2 and ϕ_2 such that they represent two different "internal helicity" states given by $k_0=+1/2$ and $-1/2$, i.e., $\phi_1=\psi(k_0=+1/2)$ and $\phi_2=\psi(k_0=-1/2)$, Eqs. (8) can be reduced to a single equation with two internal degrees of freedom, where the linear combination of $\psi(k_0=-1/2)$ and $\psi(k_0=+1/2)$ represents an eigenstate. Now, to retain the four-component nature of the Dirac spinor in Minkowski space, these two internal degrees of freedom should be assoicated with particle-antiparticle states.

Evidently this property of ϕ_1 and ϕ_2 satisfies the criterion that space, time, or conformal reflection changes one into the other. This follows from the fact that the parity operator changes the sign of k_0. Besides, since ϕ_1 and ϕ_2 are here related to particle-antiparticle states, the T operator also changes $\phi_1 \rightleftarrows \phi_2$. Again considering the Iwasawa decomposition KAN of the de Sitter group $SO(4,1)$, where K is the group $SO(4)$, the maximal compact subgroup of $SO(4,1)$, A the group $SO(1,1)$ generated by D, the dilatation operator ($D=M_{54}$), and N is a nilpotent Abelian group generated by K_i, the special conformal transformation ($K_i=M_{i5}+M_{i4}$, $i=1,2,3$), it appears that conformal reflection ($x_4 \to x_4'=-x_4$, $x_5 \to x_5'=-x_5$) also changes $\phi_1 \rightleftarrows \phi_2$ when the "internal helicity" $k_0=\pm 1/2$ and $-1/2$ are taken to correspond to the states ϕ_1 and ϕ_2, respectively. Thus, the doublet of massive spinors with spatial extension of their structures with the above properties can represent an $O(4,2)$ spinor. In view of this, we can avoid the other representations of the operator $K_0=\frac{1}{2}(a^+a-b^+b)$ of the $SO(4,1)$ group, except those with eigenvalues $k_0=\pm 1/2$, since these are the only eigenvalues that can be fully extended to $SO(4,2)$ and remain irreducible under $SO(4,1)$. Thus, the "internal helicity" states can be related to the fermion number of the massive particle when described as an extended body, since this can take the unique values $k_0=+1/2$ and $-1/2$ only.

This description of a charged and massive fermion as an extended object finds its relevance from earlier papers (Bandypadhyay, 1973, 1974), where it was shown that the charge and mass of an elementary fermion (lepton) can be taken to be of dynamical origin. There it was shown that a charged lepton such as e^- (μ^-) can be represented by ($\nu_e s$) ($\nu_\mu s$), where s is the system of photons interacting weakly with the two-component massless $\nu_e(\nu_\mu)$ at n space-time points within a quantized domain and these nonlocal interactions can give rise to the charge and mass of an e^- (μ^-)

as well as two more components necessary for a four-component spinor. The two additional componets arise here from the form factor, which describes the internal spatial extension r such that $r \to -r$ gives rise to particle and antiparticle states. These internal orientations actually correspond to the two k_0 values depicting fermion number. In this picture the quantization of charge is related with the quantization of space-time.

To find the localization region of such an extended object given by $SO(4,1)$ group structure, we here use Mackey's theory of imprimitivity. In this connection we briefly mention two main results of the Mackey theory (1949, 1952, 1953, 1958).

Let H be a closed subgroup of the lie group $\mathscr{G}$ (countable at infinity) L the unitary representation of H on Hilbert space $\mathscr{H}$, and U^L the corresponding unitarily induced representation of $\mathscr{G}$, and let $\mathscr{G}/H$ be the left coset space considered as a $\mathscr{G}$-transformation space. Then every unitarily induced representation U^L of $\mathscr{G}$ gives rise to a canonical system of imprimitivity E^L defined by

$$(E^L(\Delta)f)(x) = \begin{matrix} f(x) & \text{if } xH \in \Delta \\ 0 & \text{if } xH \notin \Delta \end{matrix} \quad \begin{pmatrix} x \in \mathscr{G} \\ f \in H^L \end{pmatrix} \tag{8}$$

Mackey's imprimitivity theorem states:

If there exists a transitive system of imprimitivity E for a given representation U of $\mathscr{G}$ based on $\mathscr{G}/H$, then there is a unitary representation L of H unique to within unitary equivalence, such that E is unitarily equivalent to the canonical system of imprimitivity for U^L.

This suggests that there is no system of imprimitivity based on $\mathscr{G}/H$ other than the canonical one and it exists whenever the considered representation U of $\mathscr{G}$ appears to be unitarily induced by a unitary representation of H. Thus, a particle is localizable in $\Delta \subset \mathscr{G}/H$ if the corresponding representation U of $\mathscr{G}$ is unitarily induced by a unitary representation of H.

In our approach a relativistic massive spinor with spatial extension is described by the unitary irreducible representation U^L of the de Sitter group $SO(4,1)$. If there exists a subgroup H of $SO(4,1)$ such that the obtained representation of $SO(4,1)$ appears to be unitarily induced by a unitary representation of H, the corresponding particle is localized in a region Δ of the space $SO(4,1)/H$. As the subgroup H we take the group MAN, where M is the group $SO(3)$, A the group $SO(1,1)$ generated by dilatation D, and N is a nilpotent Abelian group generated by K_i as discussed above. Now considering the Iwasawa decomposition of $SO(4,1)$ as KAN, K being the group $SO(4)$, we find that the extended massive particle yields the topology of a 3-sphere

$$SO(4,1)/MAN \simeq K/M \simeq S^3 \tag{9}$$

in which the particle is localized. Note that the same topology has been derived by Bayen and Niederle (1981) for a massless particle using the conformal interpretation of such particles.

4. DISCUSSION

We have shown that the non-Hermitian operator associated with the localization problem of a relativistic particle actually represents the extended nature of a massive system as well as the conservation of fermion number. In this picture fermion number is associated with the "internal helicity" or orientation of the system. For massless spinors, the conformal interpretation helps us to define a localization region that is a 3-sphere and the fermion number is given by the helicity states. In this case, though the particle may be of extended nature, the extension as such does not give rise to any physical property as in the case of a massive particle, and so for all practical purposes massless particles will appear as point particles.

For massless bosons, Bayen and Niederle have shown that a localization region given by a 3-sphere can be defined using the conformal interpretation only for spin <2. But in case of spin 1, the unitary irreducible representations $\Pi^{+}_{1,1}$, $\Pi^{-}_{1,-1}$, $\Pi^{0}_{1,0}$ appear as subrepresentations of the nonunitary representation induced by a representation $\alpha^{c}D^{1}$ of MAN, where $\Pi^{0}_{1,0}$ appears as an extension of $\Pi^{+}_{1,1}\oplus\Pi^{-}_{1,-1}$. Thus, the spin-1 massless particle can only be localized in a generalized sense. In view of these difficulties and the fact that in our above formalism bosons cannot be accommodated, we argue that all bosons are composite states.

Finally, we point out that in our formalism, fermion number is a globally conserved quantum number and since this is linked with the localization property of a relativistic particle, any fermion number-nonconserving process will be in contradiction with relativistic invariance. An important consequence of this is that a magnetic monopole is not likely to exist. This is due to the fact that when a fermion passes through the core of the monopole, the fermion is transformed into an antifermion, as demanded by the conservation of angular momentum (Ellis, 1982). From our above analysis, however, we see that this cannot occur in a Lorentz-invariant way.

REFERENCES

Angelopoulos, E., Bayen, F., and Flato, M. (1974). *Physica Scripta*, **9**, 173.
Bandyopadhyay, P. (1973). *International Journal of Theoretical Physics*, **8**, 323.
Bandyopadhyay, P. (1974). *International Journal of Theoretical Physics*, **10**, 175.
Barut, A. O., and Bohm, A. (1970). *Journal of Mathematical Physics*, **11**, 2938.
Bayen, F., and Niederle, J. (1981). *Czechoslovakian Journal of Physics B*, **31**, 1317.
Dixmier, T. (1961). *Bulletin Societé Mathematique*, **89**, 9.
Ellis, J. (1982). SLAC publication 3003.

Kalnay, A. J. (1970). *Physical Review D*, **1**, 1092.
Kalnay, A. J. (1971). *Physical Review D*, **3**, 2357.
Kalnay, A. J., and Toledo, B. P. (1967). *Nuovo Cimento* **48**, 997.
Mackey, G. W. (1949). *Proceedings of the National Academy of Sciences of the USA* **35**, 537.
Mackey, G. W. (1952). *Annals of Mathematics*, **55**, 101.
Mackey, G. W. (1953). *Annals of Mathematics*, **58**, 845.
Mackey, G. W. (1958). *Acta Mathematica*, **99**, 265.
Newton, T. D., and Wigner, E. P. (1949). *Review of Modern Physics*, **21**, 400.
Roman, P., Aghassi, J. J., and Huddleston, P. L. (1972). *Journal of Mathematical Physics*, **13**, 1852.
Wightman, A. S. (1962). *Review of Modern Physics*, **34**, 845.